2

Fredholm Integral Equations

Structure

2.1. Introduction. This chapter contains definitions and various types for Fredholm integral equations, various methods to solve Fredholm integral equations of first and second kind. Resolvent kernels are used to solve Fredholm integral equations.

2.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Boundary value problem reduced to Fredholm integral equations.

(ii) Method of successive substitution to solve Fredholm integral equation of second kind.

(iii) Method of successive approximation to solve Volterra integral equation of second kind.

(iv) Iterated kernel and Neumann series for Fredholm equations.

2.1.2. Keywords. Fredholm Integral Equations, Successive Approximations, Iterated Kernels.

2.2. Fredholm Integral equation. A Fredholm integral equation is of the type

$$h(x)\,u(x) = f(x) + \int_a^b K(x,\ \xi)\ u(\xi)\ d\xi \ \text{ for all } x \in [a, b]$$

that is, b(x) = b in this case or we can say that in Fredholm integral equation both lower and upper limits are constant.

(i) If h(x) = 0, the above equation reduces to :

$$-\ f(x) = \int_a^b K(x,\ \xi)\ u(\xi)\ d\xi$$

This equation is called **Fredholm integral equation of first kind**.

(ii) If h(x) = 1, the above equation becomes :

$$u(x) = f(x) + \int_a^b K(x\ ,\ \xi)\ u(\xi)\ d\xi$$

This equation is called **Fredholm integral equation of second kind**.

2.2.1. Example. Reduce the boundary value problem to Fredholm equation,

$$y'' + xy = 1,\ y(0) = 0,\ y(1) = 0.$$

Solution. Given boundary value problem is

$$y'' = 1 - xy \tag{1}$$

Integrating over 0 to x,

$$y'(x) = x - \int_0^x \xi\ y(\xi)\ d\xi + c_1$$

Again integrating over 0 to x,

$$y(x) = \left[\frac{x^2}{2}\right]_0^x - \int_0^x (x-\xi)\ \xi\ y(\xi)\ d\xi + c_1 x + c_2 \tag{2}$$

where c_1 and c_2 are constants to be determined by boundary value conditions.

Using y(0) = 0 in (2), we get

$$0 = 0\ -\ 0 + 0 + c_2 \qquad \Rightarrow \qquad c_2 = 0$$

So, (2) becomes

$$y(x) = \frac{x^2}{2} - \int_0^x (x-\xi)\ \xi\ y(\xi)\ d\xi + c_1 x \tag{3}$$

Now, using y(1) = 0 in (3), we get

$$0 = \frac{1}{2} - \int_0^1 (1-\xi)\ \xi\ y(\xi)\ d\xi + c_1$$

$$\Rightarrow \qquad c_1 = \int_0^1 \xi(1-\xi)\ y(\xi)\ d\xi - \frac{1}{2} \qquad (4)$$

Putting value of c_1 in (3), we get

$$y(x) = \frac{x^2}{2} - \int_0^x (x-\xi)\ \xi\ y(\xi)\ d\xi + x\int_0^1 \xi(1-\xi)\ y(\xi)\ d\xi - \frac{x}{2}$$

or $$y(x) = \frac{x^2}{2} - \frac{x}{2} - \int_0^x \xi(x-\xi)\ y(\xi)\ d\xi + x\int_0^1 \xi(1-\xi)\ y(\xi)\ d\xi$$

To express this in standard form, we split the second integral into two integrals, as follows

$$y(x) = \frac{x^2}{2} - \frac{x}{2} - \int_0^x \xi(x-\xi)\ y(\xi)\ d\xi + x\int_0^x \xi(1-\xi)\ y(\xi)\ d\xi + x\int_x^1 \xi(1-\xi)\ y(\xi)\ d\xi$$

or $$y(x) = \frac{x^2}{2} - \frac{x}{2} + \int_0^x (x - x\xi - x + \xi)\ \xi\ y(\xi)\ d\xi + x\int_x^1 (1-\xi)\ \xi\ y(\xi)\ d\xi$$

or $$y(x) = \frac{x^2}{2} - \frac{x}{2} + \int_0^x \xi(1-x)\xi\ \ y(\xi)\ d\xi + \int_x^1 \xi(1-\xi)x\ \ y(\xi)\ d\xi$$

or $$y(x) = f(x) + \int_0^1 \xi\, K(x,\xi)\ y(\xi)\ d\xi \quad \text{where } f(x) = \frac{x^2}{2} - \frac{x}{2}$$

and $$K(x, \xi) = \begin{cases} (1-x)\xi & if \quad 0 \le \xi \le x \\ (1-\xi)x & if \quad x \le \xi \le 1 \end{cases}.$$

Hence the solution.

2.2.2. Example. Reduce the boundary value problem,

$$y'' + A(x)\, y' + B(x)y = g(x),\ a \le x \le b,\ y(a) = c_1,\ y(b) = c_2$$

to a Fredholm integral equation.

Solution : Given differential equation is

$$y'' + A(x)\, y' + B(x)\, y = g(x)$$

$$\Rightarrow \qquad y'' = -A(x)\, y' - B(x)y + g(x) \qquad (1)$$

Integrating w.r.t. x from a to x, we get

$$\frac{dy}{dx} = -\int_a^x A(\xi)\ y'(\xi)\ d\xi - \int_a^x B(\xi)\ y(\xi)\ d\xi + \int_a^x g(\xi)\ d\xi + \alpha_1$$

$$\Rightarrow \qquad \frac{dy}{dx} = -\left[A(\xi)\, y(\xi)\ \right]_a^x + \int_a^x A'(\xi)\, y(\xi)\, d\xi - \int_a^x B(\xi)\, y(\xi)\, d\xi + \int_a^x g(\xi)\, d\xi + \alpha_1$$

$$\Rightarrow \qquad \frac{dy}{dx} = \int_a^x [A'(\xi) - B(\xi)]\, y(\xi) d\ (\xi) + \int_a^x g(\xi)\, d\xi - A(x)\, y(x) + A(a)\ c_1 + \alpha_1$$

Again integrating over a to x,

$$y(x) = \int_a^x (x-\xi)[A'(\xi) - B(\xi)]\, y(\xi)\, d\xi + \int_a^x (x-\xi)\, g(\xi)\, d\xi - \int_a^x A(\xi)\, y(\xi)\, d\xi + (x-a)[\alpha_1 + A(a)\ \ c_1] + \alpha_2 \tag{2}$$

Applying first boundary condition, $y(a) = c_1$, we get $\alpha_2 = c_1$

Again applying second boundary condition, $y(b) = c_2$, we have

$$c_2 = \int_a^b (b-\xi)[A'(\xi) - B(\xi)]\, y(\xi)\, d\xi + \int_a^b (b-\xi)\, g(\xi)\, d\xi - \int_a^b A(\xi)\, y(\xi)\, d\xi + (b-a)[\alpha_1 + c_1 A(a)] + c_1$$

$$\Rightarrow \alpha_1 + c_1 A(a) = \frac{1}{b-a}\left[c_2 - c_1 - \int_a^b \left[(b-\xi)\{A'(\xi) - B(\xi)\} - A(\xi)\right]\, y(\xi)\, d\xi - \int_a^b (b-\xi)\, g(\xi)\, d\xi \right]$$

$$\Rightarrow \begin{aligned} a_1 + c_1 A(a) = \frac{1}{b-a}\Bigg\{ & c_2 - c_1 - \int_a^x \left[(b-\xi)\{A'(\xi) - B(\xi)\} - A(\xi)\right]\, y(\xi)\, d\xi \\ & - \int_x^b \left[(b-\xi)\{A'(\xi) - B(\xi)\} - A(\xi)\right]\, y(\xi)\, d\xi - \int_a^b (b-\xi)\, g(\xi)\, d\xi \Bigg\} \end{aligned}$$

Putting this value of $\alpha_1 + c_1 A(a)$ in (2), we obtain

$$y(x) = c_1 + \int_a^x (x-\xi)\, g(\xi)\, d\xi + \frac{x-a}{b-a}\left[c_2 - c_1 - \int_a^b (b-\xi)\, g(\xi)\, d\xi \right]$$

$$+ \int_a^x \left[(x-\xi)\left\{A'(\xi) - B(\xi)\right\} - A(\xi)\right] y(\xi)\, d\xi - \int_a^x \frac{x-a}{b-a}\left[(b-\xi)\left\{A'(\xi) - B(\xi)\right\} - A(\xi)\right]\, y(\xi)\, d\xi$$

$$- \frac{x-a}{b-a}\int_x^b \left[(b-\xi)\left\{A'(\xi) - B(\xi)\right\} - A(\xi)\right] y(\xi)\, d\xi$$

or y(x) = f(x) + $\int_a^x \left[\left\{(x-\xi)-\frac{(x-a)(b-\xi)}{b-a}\right\}\{A'(\xi)-B(\xi)\}+A(\xi)\left\{-1+\frac{x-a}{b-a}\right\}\right]y(\xi)\,d\xi$

$$-\frac{x-a}{b-a}\int_x^b \left[(b-\xi)\{A'(\xi)-B(\xi)\}-A(\xi)\right]y(\xi)\,d\xi$$

Now, $(x-\xi)-\frac{(x-a)(b-\xi)}{b-a}=\frac{(x-b)(\xi-a)}{b-a}$ and $-1+\frac{x-a}{b-a}=\frac{x-b}{b-a}$

Thus, the above equation becomes

$$y(x)=f(x)+\frac{x-b}{b-a}\int_a^x [A(\xi)-(a-\xi)\{A'(\xi)-B(\xi)\}]\,y(\xi)\,d\xi\ -$$

$$\frac{x-a}{b-a}\int_a^b [A(\xi)-(b-\xi)\{A'(\xi)-B(\xi)\}]\,y(\xi)\,d\xi$$

or $\qquad$ y(x) = f(x) + $\int_a^b K(x\,,\ \xi)\,y(\xi)\,d\xi$

where f(x) = c_1 + $\int_a^x (x-\xi)\,g(\xi)\,d\xi+\frac{x-a}{b-a}\left[c_2-c_1-\int_a^b (b-\xi)\,g(\xi)\,d\xi\right]$

and $\qquad$ K(x, ξ) = $\begin{bmatrix} \frac{x-b}{b-a}\left[A(\xi)-(a-\xi)\{A'(\xi)-B(\xi)\}\right] & x\ >\ \xi \\ \frac{x-a}{b-a}\left[A(\xi)-(b-\xi)\{A'(\xi)-B(\xi)\}\right] & x\ <\ \xi \end{bmatrix}$

This completes the solution.

2.2.3. Example. Convert the Fredholm integral equation

u(x) = $\lambda\int_0^1 K(x\,,\ t)\,u(t)\,dt$ where K(x, t) = $\begin{cases} x(1-t) & 0\le x\le t \\ t(1-x) & t\le x\le 1 \end{cases}$ into the boundary value problem $u''+\lambda u=0,\ u(0)=0,\ u(1)=0.$

Solution. Write $\qquad$ u(x) = $\lambda\left[\int_0^x K(x\,,\ t)\,u(t)\,dt+\int_x^1 K(x\,,\ t)\,u(t)\,dt\right]$

$$=\lambda\left[\int_0^x t(1-x)\,u(t)\,dt+\int_x^1 x(1-t)\,u(t)\,dt\right]$$

$$= \lambda\int_0^x t(1-x)\, u(t)\, dt + \lambda\int_x^1 x(1-t)\, u(t)\, dt \qquad (1)$$

Differentiating (1), w.r.t. x and using Leibnitz formula

$$\frac{du}{dx} = \lambda\int_0^x -t\, u(t)\, dt + \lambda x(1-x)\, u(x) + \lambda\int_x^1 (1-t)\, u(t)\, dt - \lambda x(1-x)\, u(x)$$

So, $$\frac{du}{dx} = \lambda\int_0^x -t\, u(t)\, dt + \lambda\int_x^1 (1-t)\, u(t)\, dt$$

Again differentiating w.r.t. x and using Leibnitz rule :

$$\frac{d^2u}{dx^2} = \lambda\int_0^x 0.(-t)\, u(t)\, dt + \lambda(-x)\, u(x) + \lambda\int_x^1 0.(1-t)\, u(t)\, dt - \lambda(1-x)\, u(x)$$

$$= -\lambda\, u(x)$$

$$\Rightarrow \quad \frac{d^2u}{dx^2} + \lambda\, u(x) = 0$$

Also, from (1), we have, $u(0) = 0 = u(1)$

Hence the solution.

2.2.4. Exercise. Reduce the following boundary value problems to Fredholm integral equation.

1. $y'' - \lambda y = 0$, $a < x < b$, $y(a) = 0 = y(b)$

Answer. $y(x) = \lambda\int_a^b K(x, \xi)\, y(\xi)\, d\xi$ where $K(x, \xi) = \begin{cases} \dfrac{(x-b)(\xi-a)}{b-a} & \text{if} \quad a \le \xi \le x \\ \dfrac{(x-a)(\xi-b)}{b-a} & \text{if} \quad x \le \xi \le b \end{cases}$.

2. $y'' + \lambda y = 0$, $y(0) = 0$, $y(l) = 0$

Answer. $y(x) = \lambda\int_0^l K(x, \xi)\, y(\xi)\, d\xi$ where $K(x, \xi) = \begin{cases} \dfrac{\xi(l-x)}{l} & \text{if} \quad 0 \le \xi \le x \\ \dfrac{x(l-\xi)}{l} & \text{if} \quad x \le \xi \le l \end{cases}$.

3. $y'' + \lambda y = x$; $y(0) = 0$, $y'(1) = 0$

Answer. $y(x) = \frac{1}{6}(x^3 - 3x) + \lambda\int_0^1 K(x, \xi)\, y(\xi)\, d\xi$ where $K(x, \xi) = \begin{cases} x & , \quad x > \xi \\ \xi & , \quad x < \xi \end{cases}$.

4. $y'' + \lambda y = 2x + 1$, $y(0) = y'(1)$, $y'(0) = y(1)$

Answer. y(x) = f(x) + $\lambda \int_0^1 K(x\,,\,\xi)\,y(\xi)\,d\xi$ where f(x) = $\frac{1}{6}\left[2x^3+3x^2-17x-5\right]$ and

$K(x\,,\,\xi) = \begin{cases} 1+x(1-\xi) & \xi < x \\ (1-\xi)+(2-\xi)x & \xi > x \end{cases}$.

5. $y'' + \lambda y = e^x$ $y(0) = y'(0)$, $y(1) = y'(1)$.

Answer. $y(x) = e^x + \lambda \int_0^1 K(x\,,\xi)\,y(\xi)\,d\xi$ where $K(x\,,\,\xi) = \begin{cases} -(1+\xi)x & ; & x > \xi \\ -(1+x)\xi & ; & x < \xi \end{cases}$.

2.3. Solution of Fredholm Integral Equation. Consider a Fredholm integral equation of second kind

$$u(x) = f(x) + \lambda \int_a^b K(x\,,\,\xi)\,u(\xi)\,d\xi \qquad (1)$$

We define an integral operator,

$$k[\phi\,(x)] = \int_a^b K(x\,,\,\xi)\,\phi(\xi)\,d\xi$$

$k^2[\phi\,(x)] = k\left[k\left\{\phi(x)\right\}\right]$ and so on.

Then, (1) can be written as

$$u(x) = f(x) + \lambda\, k\,[u(x)].$$

2.3.1. Theorem. If the Fredholm integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x\,,\,\xi)\,u(\xi)\,d\xi \qquad (1)$$

is such that

(i) $K(x,\,\xi)$ is a non – zero real valued continuous function in the rectangle $R = I\times I$, where I = [a, b] and $\left|K(x\,,\,\xi)\right| < M$ in R .

(ii) f(x) is a non-zero real valued and continuous function on I.

(iii) λ is a constant satisfying the inequality, $|\lambda| < \frac{1}{M(b-a)}$.

Then (1) has one and only one continuous solution in the interval I and this solution is given by the absolutely and uniformly convergent series $u(x) = f(x) + \lambda\, k\,[f(x)] + \lambda^2 k^2\,[f(x)] + \ldots$ to ∞.

Proof. We prove the result by the method of successive approximation. In this method we choose any continuous function say $u_0(x)$ defined on I as the zeroth approximation.

Then the first approximation, say $u_1(x)$, is given

$$u_1(x) = f(x) + \lambda \int_a^b K(x, \xi)\, u_0(\xi)\, d\xi \qquad (2)$$

By substituting this approximation into R.H.S. of (1), we obtain next approximation, $u_2(x)$. Continuing like this, we observe that the successive approximations are determined by the recurrence formula

$$u_n(x) = f(x) + \lambda \int_a^b K(x, \xi)\, u_{n-1}(\xi)\, d\xi \qquad (3)$$

$$= f(x) + \lambda\, k\left[u_{n-1}(x)\right]$$

$$= f(x) + \lambda k\left[f(x) + \lambda k\left\{u_{n-2}(x)\right\}\right]$$

$$= f(x) + \lambda k\left[f(x)\right] + \lambda^2 k^2\left[f(x) + \lambda k\left\{u_{n-3}(x)\right\}\right]$$

Hence, $u_n(x) = f(x) + \lambda k\left[f(x)\right] + \lambda^2 k^2\left[f(x)\right] + \ldots + \lambda^{n-1} k^{n-1}\left[f(x)\right] + R_n(x)$,

where $R_n(x) = \lambda^n\, k^n[u_0(x)]$.

As $u_0(x)$ is continuous, it is bounded that is, $|u_0(x)| \le U$ in I

Now, $$|R_n(x)| = |\lambda|^n \left[\int_a^b K(x, t) \int_a^b K(t, t_1) \ldots \int_a^b K(t_{n-2}, t_{n-1})\, u_0(t_{n-1})\, dt_{n-1} \ldots dt\right]$$

$$\le |\lambda|^n\; M^n\; U\; (b-a)^n$$

$$= U\left[|\lambda|\; M\,(b-a)\right]^n \to 0 \text{ as } n \to \infty \quad \left(\text{Since}, \; |\lambda| < \frac{1}{M(b-a)}\right)$$

$$\Rightarrow \qquad \lim_{n \to \infty} R_n(x) = 0$$

Thus, $\lim_{n \to \infty} u_n(x) = u(x) = f(x) + \lambda\, k\, f(x) + \lambda^2 k^2\, f(x) + \ldots \text{ to } \infty$

This can be easily verified by the virtue of M – test that the above series is absolutely and uniformly convergent in I.

Uniqueness. Let $v(x)$ be another solution of given integral equation then by choosing $u_0(x) = v(x)$, we get

$$u_n(x) = v(x) \text{ for all } n$$

$$\Rightarrow \qquad \lim_{n \to \infty} u_n(x) = v(x) \Rightarrow u(x) = v(x).$$

This completes the proof.

2.3.2. Example. Find the first two approximation of the solution of Fredholm integral equation.

$$u(x) = 1 + \int_0^1 K(x, \xi)\, u(\xi)\, d\xi \text{ where } K(x, \xi) = \begin{cases} x & 0 \le x \le \xi \\ \xi & \xi \le x \le 1 \end{cases}.$$

Solution. Let $u_0(x) = 1$ be the zeroth approximation. Then first approximation is given by

$$u_1(x) = 1 + \int_0^1 K(x, \xi)\, u_0(\xi)\, d\xi$$

$$= 1 + \int_0^x K(x, \xi)\, d\xi + \int_x^1 K(x, \xi)\, d\xi \qquad = 1 + \int_0^x \xi\, d\xi + \int_x^1 x\, d\xi$$

$$= 1 + \frac{x^2}{2} + x(1-x) = 1 + x - \frac{x^2}{2}$$

Now, $$u_2(x) = 1 + \int_0^1 K(x, \xi)\, u_1(\xi)\, d\xi$$

$$= 1 + \int_0^1 K(x, \xi)\left(1+\xi-\frac{\xi^2}{2}\right) d\xi$$

$$= 1 + \int_0^x \xi \left(1+\xi-\frac{\xi^2}{2}\right) d\xi + x\int_x^1 \left(1+\xi-\frac{\xi^2}{2}\right) d\xi$$

$$= 1 + \frac{4}{3}x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}\ .$$

2.4. Resolvent kernel for Fredholm integral equation. Consider the Fredholm integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x, \xi)\, u(\xi)\, d\xi \qquad (1)$$

The iterated kernels are defined by $K_1(x, \xi) = K(x, \xi)$, and

$$K_{n+1}(x, \xi) = \int_a^b K(x, t)\, K_n(t, \xi)\, dt\ , \quad n = 1, 2, 3, \ldots$$

and the solution of (1) is given by :

$$u(x) = f(x) + \lambda \int_a^b R(x, \xi : \lambda)\, f(\xi)\, d\xi$$

where $$R(x, \xi : \lambda) = K_1 + \lambda K_2 + \lambda^2 K_3 + \ldots \text{ to } \infty \qquad = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x, \xi)$$

2.4.1. Neumann series. The infinite series $K_1 + \lambda K_2 + \lambda^2 K_3 + \ldots\ldots$ is called Neumann series.

2.4.2. Resolvent Kernel. The function $R(x, \xi : \lambda)$ is called Resolvent Kernel.

2.4.3. Example. Obtain the Resolvent kernel associated with the kernel $K(x, \xi) = 1 - 3x\xi$ in the interval (0, 1) and solve the integral equation $u(x) = 1 + \lambda \int_0^1 (1 - 3x\xi)\, u(\xi)\, d\xi$.

Solution. Here, $K(x, \xi) = 1-3x\xi$. We know that the iterated kernels are given by the relation,

$$K_1(x, \xi) = K(x, \xi)$$

and $$K_{n+1}(x, \xi) = \int_a^b K(x, t)\, K_n(t, \xi)\, dt$$

Therefore, $$K_1(x, \xi) = 1-3x\xi$$

and $$K_2(x, \xi) = \int_0^1 K(x, t)\, K_1(t, \xi)\, dt$$

$$= \int_0^1 (1-3xt)\,(1-3t\xi)\, dt$$

$$= \int_0^1 (1-3t\xi-3xt+9xt^2\xi)\, dt$$

$$= \left[t-\frac{3t^2\xi}{2}-\frac{3xt^2}{2}+3xt^3\xi\right]_0^1$$

$$= 1-\frac{3}{2}\xi-\frac{3}{2}x+3x\xi$$

$$K_3(x, \xi) = \int_0^1 K(x, t)\, K_2(t, \xi)\, dt$$

$$= \int_0^1 (1-3xt)\left(1-\frac{3}{2}t-\frac{3}{2}\xi+3t\xi\right) dt$$

$$= \frac{1}{4}(1-3x\xi) \text{ (on solving)}$$

$$K_4(x, \xi) = \int_0^1 K(x, t)\, K_3(t, \xi)\, dt$$

$$= \frac{1}{4}\int_0^1 (1-3xt)\,(1-3t\xi)\, dt$$

$$= \frac{1}{4}\left[1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right]$$

The Resolvent Kernel $R(x, \xi : \lambda)$ is given by

$$R(x, \xi : \lambda) = K_1 + \lambda K_2 + \lambda^2 K_3 + \lambda^4 K_4 + \ldots$$

$$= (1-3x\xi)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)+\frac{\lambda^2}{4}(1-3x\xi)+\frac{\lambda^3}{4}\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)+\ldots$$

$$= (1-3x\xi)\left(1+\frac{\lambda^2}{4}\right)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)\left(1+\frac{\lambda^2}{4}\right)+\ldots$$

$$= \left(1+\frac{\lambda^2}{4}+...\right)\left[(1-3x\xi)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)\right]$$

$$= \left(\frac{1}{1-\frac{\lambda^2}{4}}\right)\left[(1-3x\xi)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)\right]$$

$$= \left(\frac{4}{4-\lambda^2}\right)\left[(1-3x\xi)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)\right]$$

which provides the required result.

We know that the solution of an integral equation

$$u(x) = f(x) + \lambda\int_a^b K(x\,,\,\xi)\;u(\xi)\,d\xi \text{ is given by}$$

$$u(x) = f(x) + \lambda\int_a^b R(x\,,\,\xi:\lambda)\,f(\xi)\,d\xi$$

Here, $K(x,\ \xi) = (1-3x\xi)$. Then,

$$R(x,\ \xi\ :\ \lambda) = \left(\frac{4}{4-\lambda^2}\right)\left[(1-3x\xi)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)\right]$$

Thus, the solution of given integral equation is

$$u(x) = 1+\frac{4\lambda}{4-\lambda^2}\int_0^1\left[(1-3x\xi)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)\right].1d\xi$$

$$= 1+\frac{4\lambda}{4-\lambda^2}\left[\xi-3x\frac{\xi^2}{2}+\lambda\left(\xi-\frac{3\xi^2}{4}-\frac{3x\xi}{2}+\frac{3x\xi^2}{2}\right)\right]_0^1$$

$$= 1+\frac{4\lambda}{4-\lambda^2}\left[1-\frac{3x}{2}+\lambda\left(1-\frac{3}{4}-\frac{3x}{2}+\frac{3x}{2}\right)\right]$$

$$= 1+\frac{4\lambda}{4-\lambda^2}\left(1-\frac{3x}{2}+\frac{\lambda}{4}\right) = \frac{4+4\lambda-6x\lambda}{4-\lambda^2},\ \lambda\neq\pm 2$$

This is the required solution of given integral equation.

2.4.4. Exercise. Determine the Resolvent Kernel associated with $K(x,\ \xi) = x\xi$ in the interval (0, 1) in the form of a power series in λ.

Answer. $R(x,\ \xi\ :\ \lambda) = \dfrac{3}{3-\lambda}x\xi\ ,\ \ |\lambda| < 3$

2.4.5. Exercise. Solve the following integral equations by finding the resolvent kernel:

1. $u(x) = f(x) + \lambda\int_0^1 e^{(x-\xi)}u(\xi)\, d\xi$

Answer. $u(x) = f(x) + \frac{\lambda}{1-\lambda}\int_0^1 e^{(x-\xi)}f(\xi)\, d\xi$.

2. $u(x) = 1 + \lambda\int_0^1 xe^{\xi}\ u(\xi)\, d\xi$

Answer. $u(x) = 1 + \frac{\lambda x}{1-\lambda}(e-1)$

3. $u(x) = x + \lambda\int_0^1 xe^{\xi}\ u(\xi)\, d\xi$

Answer. $u(x) = x + \frac{\lambda x}{1-x}$

4. $u(x) = x + \lambda\int_0^1 x\xi\ u(\xi)\, d\xi$

Answer. $u(x) = x + \frac{\lambda x}{3-\lambda}$.

2.5. Separable kernel. A kernel $K(x, \xi)$ of an integral equation is called separable if it can be expressed in the form

$$K(x, \xi) = \sum_{i=1}^{n} a_i(x)\, b_i(\xi) = a_1(x)\, b_1(\xi) + a_2(x)\, b_2(\xi) + \ldots + a_n(x)\, b_n(\xi)$$

For example, (a) $e^{x-\xi} = e^x . e^{-\xi} = a_1(x)\, b_1(\xi)$, n = 1

(b) $x-\xi = x.1 + 1(-\xi) = a_1(x)\, b_1(\xi) + a_2(x)\, b_2(\xi)$, n = 2

(c) Similarly, $\sin(x+\xi)$, $1-3x\xi$ are separable kernels.

(d) x^{ξ}, $\sin(x\xi)$ are non – separable kernels.

2.5.1. Method to solve Fredholm integral equation of second kind with separable kernel.

Let the given integral equation be

$$u(x) = f(x) + \lambda\int_a^b K(x, \xi)\, u(\xi)\, d\xi \qquad (1)$$

where $$K(x, \xi) = \sum_{i=1}^{n} a_i(x)\, b_i(\xi) \qquad (2)$$

Thus, (1) can be written as

$$u(x) = f(x) + \lambda \int_a^b \left[\sum_{i=1}^{n} a_i(x)\, b_i(\xi) \right] u(\xi)\, d(\xi)$$

$$u(x) = f(x) + \lambda \sum_{i=1}^{n} a_i(x) \left[\int_a^b b_i(\xi)\, u(\xi)\, d(\xi) \right]$$

$$= f(x) + \lambda \left[c_1 a_1(x) + c_2 a_2(x) + \ldots + c_n a_n(x) \right] \qquad (3)$$

where $$C_k = \int_a^b b_k(\xi)\, u(\xi)\, d\xi = \int_a^b b_k(x)\, u(x)\, dx \qquad (4)$$

Here, (3) gives the solution of given Fredholm integral (1) provided the constants $c_1, c_2, \ldots, c_n$ are determined.

For this, we multiply (3) both sides by $b_i(x)$ and then integrating w.r.t. x from a to b, we find

$$\int_a^b b_i(x)\, u(x)\, dx = \int_a^b f(x)\, b_i(x)\, dx + \lambda \sum_{k=1}^{n} C_k \int_a^b b_i(x)\, a_k(x)\, dx \quad \text{for } i = 1, 2, 3, \ldots, n$$

$$\Rightarrow \qquad c_i = f_i + \lambda \sum_{k=1}^{n} \alpha_{ik} C_k \qquad (5)$$

where $$f_i = \int_a^b f(x)\, b_i(x)\, dx \ \text{ and } \ \alpha_{ik} = \int_a^b b_i(x)\, a_k(x)\, dx \qquad (6)$$

Now, from (5)

$$c_1 = f_1 + \lambda \left[\alpha_{11} c_1 + \alpha_{12} c_2 + \ldots + \alpha_{1n} c_n \right]$$

$$c_2 = f_2 + \lambda \left[\alpha_{21} c_1 + \alpha_{22} c_2 + \ldots + \alpha_{2n} c_n \right]$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$c_n = f_n + \lambda \left[\alpha_{n1} c_1 + \alpha_{n2} c_2 + \ldots + \alpha_{nn} c_n \right]$$

In matrix form, $C = F + \lambda AC$ or $(I - \lambda A) C = F$ (7)

where $$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},\ F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix},\ A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \ldots & \alpha_{nn} \end{bmatrix}$$

Let $$|I - \lambda A| = \Delta(\lambda) \qquad (8)$$

Now, we discuss the various cases

Case I. When $f(x) \neq 0$ and $F \neq 0$, that is, both integral equation as well as matrix equation are non – homogeneous. Then, (7) has a unique solution if and only if $\Delta(\lambda) \neq 0$

If $\Delta(\lambda) = 0$ for some value of λ, then (7) has no solution or infinite solutions.

Case II. When $f(x) = 0$ that is, the Fredholm integral equation is homogeneous. In this case $f_i = o$ for all i and consequently $F = 0$. Thus, (7) reduces to :

$$(I - \lambda A)C = 0 \tag{9}$$

Subcase (a). If $\Delta(\lambda) \neq 0$, then (9) has the trivial solution, $C = 0$ that is, $C_i = 0$ for all i .

Hence the (3) becomes, $u(x) = 0$ which is the solution of given integral equation.

Subcase (b). If $\Delta(\lambda_0) = 0$ for some scalar λ_0, then (9) has infinitely many solutions. Consequently, the Fredholm integral equation $u(x) = \lambda_0 \int_a^b K(x, \xi)\, u(\xi)\, d\xi$ has infinitely many solutions.

Case III. When $f(x) \neq 0$ but $F = 0$. In this case also,

$$(I - \lambda A)\, C = 0 \tag{10}$$

Subcase (a). If $\Delta(\lambda) \neq 0$, then (10) has only trivial solution $C = 0$ that is, $C_i = 0$ for all i.

Hence the required solution of given equation becomes

$$u(x) = f(x) + 0 = f(x)$$

Sub case (b). If $\Delta(\lambda_0) = 0$ for some scalar $\lambda = \lambda_0$, then (10) has infinitely many solutions, therefore the given equation, $u(x) = f(x) + \lambda_0 \int_a^b K(x, \xi)\, u(\xi)\, d\xi$ has infinitely many solutions.

2.5.2. Eigen values and Eigen functions. The values of λ for which $\Delta(\lambda) = 0$ are called eigen values (or characteristic numbers) of Fredholm integral equation. The non – trivial solution corresponding to eigen values are called eigen functions (or characteristic functions).

Remark. Separable kernels are also known as degenerate kernels.

2.5.3. Example. Solve the integral equation and discuss all its possible cases with the method of separable kernels

$$u(x) = f(x) + \lambda \int_0^1 (1 - 3x\xi)\, u(\xi)\, d\xi\,.$$

Solution. The given equation is

$$u(x) = f(x) + \lambda \int_0^1 (1 - 3x\xi)\, u(\xi)\, d\xi \tag{1}$$

$$\Rightarrow \quad u(x) = f(x) + \lambda\,[C_1 - 3xC_2] \tag{2}$$

where, $$c_1 = \int_0^1 u(\xi)\, d\xi \tag{3}$$

and $$c_2 = \int_0^1 \xi\, u(\xi)\, d\xi \tag{4}$$

c_1 and c_2 are constants to be determined.
Integrating (2), w.r.t.x over the limit 0 to 1.

$$\int_0^1 u(x)\,dx = \int_0^1 f(x)\,dx + \lambda \int_0^1 (c_1 - 3xc_2)\,dx$$

$$\Rightarrow \quad c_1 = \int_0^1 f(x)\,dx + \lambda\left(c_1 - \frac{3}{2}c_2\right) \qquad \text{[Using (3)]}$$

or $$(1-\lambda)\,c_1 + \frac{3}{2}\lambda\, c_2 = f_1 \tag{5}$$

where $$f_1 = \int_0^1 f(x)\,dx$$

Now multiplying (2) with x and integrating w.r.t. x between limits 0 and 1, we get

$$\int_0^1 x\,u(x)\,dx = \int_0^1 x\,f(x)\,dx + \lambda \int_0^1 \left(c_1 x - 3x^2 c_2\right)dx$$

or $$c_2 = f_2 + \lambda\left[c_1 \frac{x^2}{2} - x^3 c_2\right]_0^1 \qquad \text{[Using (4)]}$$

$$= f_2 + \lambda\left(\frac{c_1}{2} - c_2\right)$$

or $$-\frac{\lambda}{2}c_1 + (1+\lambda)\,c_2 = f_2 \tag{6}$$

where $f_2 = \int_0^1 x\,f(x)\,dx$

From (5) and (6), we get, $\Delta(\lambda) = \begin{vmatrix} 1-\lambda & \frac{3\lambda}{2} \\ -\frac{\lambda}{2} & 1+\lambda \end{vmatrix} = 1 - \lambda^2 + \frac{3\lambda^2}{4} = 1 - \frac{\lambda^2}{4}$

or $$\Delta(\lambda) = \frac{4-\lambda^2}{4}$$

Now, (5) and (6) can be written as

$$(I-\lambda A)\,C = f$$

where $C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, $F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$.

Also, $|I-\lambda A| = \Delta(\lambda)$.

Case I. When $f(x)\neq 0$ and $F\neq 0$ then equations (5) and (6) has a unique solution if $\Delta(\lambda)\neq 0$, that is, $\lambda \neq 2, -2$. When $\lambda = 2$ or -2, then these equations have either no solution or infinite many solutions.

(i) $\lambda = 2$

Then, (5) and (6) reduce to

$$\left.\begin{aligned} -c_1+3c_2 &= f_1 \\ -c_1+3c_2 &= f_2 \end{aligned}\right] \tag{7}$$

These equation have no solution if $f_1\neq f_2$ and have infinitely many solutions when $f_1 = f_2$, that is,

$$\int_0^1 f_1(x)\,dx = \int_0^1 x\,f(x)\,dx$$

or $$\int_0^1 (1-x)\,f(x)\,dx = 0$$

Thus, the solution of given integral equation is

$$\begin{aligned} u(x) &= f(x) + 2[c_1a_1(x) + c_2a_2(x)] \\ &= f(x) + 2[c_1.1 + c_2(-3x)] = f(x) + 2[3c_2-f_1-3xc_2] \qquad \text{[From (7)]} \\ &= f(x) + 6c_2(1-x) - 2f_1 \end{aligned}$$

or $$u(x) = f(x) + 6c_2(1-x) - 2\int_0^1 f(x)\,dx \quad \text{where } c_2 \text{ is arbitrary.}$$

(ii) $\lambda = -2$

As done above, the solution is given by

$$u(x) = f(x) - 2(1-3x)c_2 - 2\int_0^1 x\,f(x)\,dx$$

Case II. When $f(x) = 0$, $F = 0$

In this case, the equations (5) and (6) becomes :

$$\begin{aligned} (1-\lambda)\, c_1 + \frac{3\lambda}{2}\, c_2 &= 0 \\ \frac{-\lambda}{2}\, c_1 + (1+\lambda)\, c_2 &= 0 \end{aligned} \qquad (8)$$

If $\lambda \neq 2, -2$, then system has only trivial solution $c_1 = 0 = c_2$. Thus $u(x) = 0$ is the solution of given integral equation.

(i) $\lambda = 2$

Then, (8) becomes

$$-c_1 + 3c_2 = 0 \quad \Rightarrow \quad c_1 = 3c_2$$

Thus the solution of given integral equation is

$$u(x) = 0 + 2(3c_2 - 3xc_2) = 6c_2(1-x).$$

(ii) $\lambda = -2$

Then, (8) becomes

$$c_1 - c_2 = 0 \Rightarrow c_1 = c_2$$

Thus the solution is

$$u(x) = 0 - 2[c_2 - 3xc_2] = 2c_2(3x-1).$$

Case III. When $f(x) \neq 0$ and $F = 0$

If $\lambda \neq 2, -2$, the system (8) has only trivial solution $c_1 = c_2 = 0$ and therefore $u(x) = f(x)$ is the solution.

(i) $\lambda = 2$

Then $c_1 = 3c_2$ and the solution is

$$u(x) = f(x) + 2(3c_2 - 3xc_2) = f(x) + 6c_2(1-x).$$

(ii) $\lambda = -2$

Then $c_1 = c_2$ and the solution is

$$u(x) = f(x) - 2[c_2 - 3xc_2] = f(x) - 2c_2(1-3x)$$

This completes the solution.

2.5.4. Example : Find the eigen values and eigen functions of the integral equation

$$u(x) = \lambda \int_0^{2\pi} \sin(x+t)\, u(t)\, dt\ .$$

Answer. Eigen values are $\lambda = \pm\frac{1}{\pi}$. For $\lambda = \frac{1}{\pi}$, eigen function is u(x) = A(sinx + cosx), where A = $\frac{c_1}{\pi}$ and for $\lambda = -\frac{1}{\pi}$, eigen function is u(x) = B(sinx − cosx), where B = $\frac{c_2}{\pi}$.

2.5.5. Exercise. Solve the integral equations by the method of degenerate kernel:

1. $u(x) = x + \lambda\int_0^1 (xt^2 + x^2 t)\, u(t)\, d(t)$

Answer. $u(x) = \dfrac{(240-60\lambda)\, x + 80\lambda x^2}{240 - 120\lambda - \lambda^2}$.

2. $u(x) = e^x + \lambda\int_0^1 2e^x e^t\, u(t)\, dt$

Answer. $u(x) = \dfrac{e^x}{1-\lambda(e^2-1)}$.

2.6. Symmetric kernel. The kernel K(x, ξ) of an integral equation is said to be symmetric if

$$K(x, \xi) = K(\xi, x) \text{ for all } x \text{ and } \xi.$$

2.6.1. Orthogonality. Two functions $\phi_1(x)$ and $\phi_2(x)$ continuous on an interval (a, b) are said to be orthogonal if $\int_a^b \phi_1(x)\, \phi_2(x)\, dx = 0$.

2.6.2. Theorem. For the Fredholm integral equation $y(x) = \lambda\int_a^b K(x, \xi)\, y(\xi)\, d\xi$ with symmetric kernel, prove that :

(i) The eigen functions corresponding to two different eigen values are orthogonal over (a, b).

(ii) The eigen values are real.

Proof. (i) Let λ_1 and λ_2 be two different eigen values of given integral equation

$$y(x) = \lambda\int_a^b K(x, \xi)\, y(\xi)\, d\xi \tag{1}$$

w.r.t. eigen functions $y_1(x)$ and $y_2(x)$. We have to show that

$$\int_a^b y_1(x)\, y_2(x)\, dx = 0 \tag{2}$$

By definition we have,

$$y_1(x) = \lambda_1 \int_a^b K(x, \xi)\, y_1(\xi)\, d\xi \qquad (3)$$

$$y_2(x) = \lambda_2 \int_a^b K(x, \xi)\, y_2(\xi)\, d\xi \qquad (4)$$

Multiplying (3) by $y_2(x)$ and then integrating w.r.t. x over the interval a to b, we find

$$\int_a^b y_1(x)\, y_2(x)\, dx = \lambda_1 \int_a^b y_2(x) \left[\int_a^b K(x, \xi)\, y_1(\xi) d\xi \right] dx$$

Interchanging the order of integration

$$\int_a^b y_1(x)\, y_2(x)\, dx = \lambda_1 \int_a^b y_1(\xi) \left[\int_a^b K(x, \xi)\, y_2(x) dx \right] d\xi$$

$$= \lambda_1 \int_a^b y_1(\xi) \left[\int_a^b K(\xi, x)\, y_2(x)\, dx \right] d\xi \text{ [Since } K(x, \xi) = K(\xi, x)]$$

$$= \lambda_1 \int_a^b y_1(\xi) \frac{y_2(\xi)}{\lambda_2} d\xi \qquad \text{[By (4)]}$$

$$= \frac{\lambda_1}{\lambda_2} \int_a^b y_1(x)\, y_2(x)\, dx$$

$$\Rightarrow \quad \left(1 - \frac{\lambda_1}{\lambda_2}\right) \int_a^b y_1(x)\, y_2(x)\, dx = 0$$

$$\Rightarrow \quad \int_a^b y_1(x)\, y_2(x)\, dx = 0, \qquad (\lambda_1 \neq \lambda_2)$$

(ii) If possible, we assume on the contrary that there is an eigen value λ_0 (say) which is not real.

So, $\lambda_0 = \alpha_0 + i\beta_0,\ \beta_0 \neq 0$ (5)

where α_0 and β_0 are real.

Let $y_0(x) \neq 0$ be the corresponding eigen function. Then

$$y_0(x) = \lambda_0 \int_a^b K(x, \xi)\, y_0(\xi)\, d\xi \qquad (6)$$

We claim that the eigen function $y_0(x)$ corresponding to a non real eigen value λ_0 is not real valued. If $y_0(x)$ is real valued, then separating the real and imaginary parts in (6), we get

$$y_0(x) = \alpha_0 \int_a^b K(x, \xi)\ y_0(\xi)\ d\xi \qquad (7)$$

and
$$0 = \beta_0 \int_a^b K(x, \xi)\ y_0(\xi)\ d\xi \qquad (8)$$

$$\Rightarrow \quad \int_a^b K(x, \xi)\ y_0(\xi)\ d\xi = 0, \ (\beta_0 \neq 0)$$

Hence from (7), we get $y_0(x) = 0$, a contradiction. Thus $y_0(x)$ cannot be a real valued function.

Let us consider

$$y_0(x) = \alpha(x) + i\beta_0(x),\ \beta(x) \neq 0 \qquad (9)$$

Changing i to $-$i in (6), we obtain

$$\overline{y_0(x)} = \overline{\lambda_0} \int_a^b K(x, \xi)\ \overline{y_0(\xi)}\ d\xi \qquad (10)$$

This shows that $\overline{\lambda_0}$ is an eigen value with corresponding eigen function $\overline{y_0(x)}$. Since λ_0 is non – real by assumption. So λ_0 and $\overline{\lambda_0}$ are two different eigen values. Thus by part (i), we have

$$\int_a^b y_0(x)\ \overline{y_0(x)}\ dx = 0$$

$$\Rightarrow \quad \int_a^b |y_0(x)|^2\ dx = 0$$

$$\Rightarrow \quad \int_a^b |\alpha(x) + i\beta(x)|^2\ dx = 0$$

$$\Rightarrow \quad \int_a^b \left([\alpha(x)]^2 + [\beta(x)]^2\right)\ dx = 0$$

$$\Rightarrow \quad \alpha(x) = \beta(x) = 0$$

$$\Rightarrow \quad y_0(x) = 0,$$

a contradiction because eigen functions are non – zero. This contradiction shows that our assumption that λ_0 is not real is wrong. Hence λ_0 must be real.

This completes the proof.

2.6.3. Fredholm Resolvent kernel expressed as a ratio of two series in λ.

Consider the Fredholm integral equation

$$u(x) = f(x) + \lambda\int_a^b K(x,\xi)\; u(\xi)\; d\xi \qquad (1)$$

The resolvent kernel of (1) is also given by

$$R(x, \xi : \lambda) = \frac{D(x, \xi : \lambda)}{D(\lambda)} \qquad [D(\lambda) \neq 0]$$

where, $D(x, \xi : \lambda) = K(x, \xi) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \lambda^n\; B_n(x, \xi)$

and $D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \lambda^n\; c_n$

where

$$B_n(x, \xi) = \underbrace{\int_a^b\int_a^b \cdots\cdots \int_a^b}_{n \text{ times}} \begin{vmatrix} K(x,\xi) & K(x,t_1) & \cdots & K(x,t_n) \\ K(t_1,\xi) & K(t_1,t_1) & \cdots & K(t_1,t_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(t_n,\xi) & K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1\, dt_2 \ldots\ldots dt_n$$

and

$$c_n = \underbrace{\int_a^b\int_a^b \cdots\cdots \int_a^b}_{n \text{ times}} \begin{vmatrix} K(t_1,t_1) & K(t_1,t_2) & \cdots & K(t_1,t_n) \\ K(t_2,t_1) & K(t_2,t_2) & \cdots & K(t_2,t_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(t_n,t_1) & K(t_n,t_2) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1\, dt_2 \ldots\ldots dt_n$$

Note that determinant in c_n is obtained by just removing first row and first column from the determinant in B_n.

2.6.4. Fredholm Determinant. $D(x, \xi : \lambda)$ is called Fredholm minor and $D(\lambda)$ is called Fredholm determinant.

Remark.

1. After finding the resolvent kernel $R(x, \xi : \lambda)$ the solution of given integral equation is given by

$$u(x) = f(x) + \lambda\int_a^b R(x, \xi : \lambda)\; f(\xi)\; d\xi$$

2. This method cannot be used when $\lambda = 1$.

2.6.5. Example. Using the Fredholm determinant, find the resolvent kernel of

$$K(x, \xi) = 2x - \xi, \qquad 0 \le x \le 1,\ 0 \le \xi \le 1.$$

Solution. Here the kernel is

$$K(x,\ \xi) = 2x - \xi \qquad (1)$$

The resolvent kernel $R(x,\ \xi : \lambda)$ is given by

$$R(x,\ \xi : \lambda) = \frac{D(x\,,\xi:\lambda)}{D(\lambda)}\,,\ D(\lambda) \neq 0 \qquad (2)$$

where $$D(x,\ \xi : \lambda) = K(x,\ \xi) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\,\lambda^n\ B_n(x\,,\xi)$$

and $$D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\,\lambda^n\ c_n \qquad (3)$$

where $$B_n = \int_a^b\int_a^b \ldots\ldots \int_a^b \begin{vmatrix} K(x\,,\xi) & K(x\,,t_1) & \cdots & K(x\,,t_n) \\ K(t\ \ ,\xi) & K(t_1\,,t_1) & \cdots & K(t_1\,,t_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(t_n\,,\xi) & K(t_n\,,t_1) & \cdots & K(t_n\,,t_n) \end{vmatrix} dt_1\,dt_2\ldots\ldots dt_n$$

and $$\int_a^b\int_a^b \ldots\ldots \int_a^b \begin{vmatrix} K(t_1\,,t_1) & K(t_1\,,t_2) & \cdots & K(t_1\,,t_n) \\ K(t_2\,,t_1) & K(t_2\,,t_2) & \cdots & K(t_2\,,t_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(t_n\,,t_1) & K(t_n\,,t_2) & \cdots & K(t_n\,,t_n) \end{vmatrix} dt_1\,dt_2\ldots\ldots dt_n$$

Therefore, $$B_1(x,\ \xi) = \int_0^1 \begin{vmatrix} 2x-\xi & 2x-t_1 \\ 2t_1-\xi & 2t_1-t_1 \end{vmatrix} dt_1$$

$$= \int_0^1 \left(2xt_1 - \xi t_1 - 4xt_1 + 2t_1^2 + 2x\xi - \xi t_1\right) dt_1$$

$$= \int_0^1 \left(-2xt_1 - 2\xi t_1 + 2t_1^2 + 2x\xi\right) dt_1$$

$$B_1(x,\ \xi) = -x - \xi + \frac{2}{3} + 2x\xi$$

$$B_2(x,\ \xi) = \int_0^1\int_0^1 \begin{vmatrix} 2x-\xi & 2x-t_1 & 2x-t_2 \\ 2t_1-\xi & 2t_1-t_1 & 2t_1-t_2 \\ 2t_2-\xi & 2t_2-t_1 & 2t_2-t_2 \end{vmatrix} dt_1dt_2$$

which on solving gives, $B_2(x,\ \xi) = 0$

In general $B_n(x,\ \xi) = 0$ for all $n \geq 2$

Now, $$c_1 = \int_0^1 \left(2t_1 - t_1\right) dt_1 = \frac{1}{2}$$

$$c_2 = \int_0^1\int_0^1 \begin{vmatrix} 2t_1 - t_1 & 2t_1 - t_2 \\ 2t_2 - t_1 & 2t_2 - t_2 \end{vmatrix} dt_1 dt_2 = \frac{1}{3}$$

Now, since $B_n = 0$ for all $n \geq 2$

$\Rightarrow \quad c_n = 0$ for all $n \geq 3$

Thus, from (3), we get

$$D(x, \xi : \lambda) = (2x - \xi) + (-1)\ \lambda \left(2\xi x - x - \xi + \frac{2}{3}\right)$$

$$= 2x - \xi + \lambda \left(x + \xi - 2x\xi - \frac{2}{3}\right)$$

$$D(\lambda) = 1 + (-1)^1\ \lambda\, c_1 + \frac{(-1)^2}{2!}\ \lambda\, c_2 = 1 - \frac{\lambda}{2} + \frac{\lambda^2}{6}$$

Hence the resolvent kernel is given by :

$$R(x, \xi : \lambda) = \frac{(2x - \xi) + \lambda\left((x + \xi - 2x\xi - \frac{2}{3}\right)}{1 - \frac{\lambda}{2} + \frac{\lambda^2}{6}}$$

Hence the solution.

2.6.6. Exercise. Using Fredholm determinant, find the resolvent kernel of $K(x, \xi) = 1 + 3x\xi$.

Answer. $$R(x,\xi:\lambda) = \frac{(1 + 3x\xi) - \lambda\left(1 + 3\xi x - \frac{3\xi}{2} - \frac{3x}{2}\right)}{1 - 2\lambda + \frac{\lambda^2}{4}}.$$

2.7. Check Your Progress.

1. Solve the following integral equations by finding the resolvent kernel:

$$u(x) = f(x) + \lambda \int_0^1 e^{a(x^2 - \xi^2)}\ u(\xi)\, d\xi$$

Answer. $u(x) = f(x) + \dfrac{\lambda}{1-\lambda} \displaystyle\int_0^1 e^{a(x^2 - \xi^2)}\ f(\xi)\, d\xi$.

2. Solve the integral equations by the method of degenerate kernel:

$$u(x) = x + \lambda \int_0^1 (1 + x + t)\, u(t)\, dt$$

Answer. $u(x) = x + \dfrac{\lambda}{12 - 24\lambda - \lambda^2}[10 + (6 + \lambda)x]$.

2.8. Summary. In this chapter, various methods like successive approximations, successive substitutions, resolvent kernel are discussed to solve a Fredholm integral equation. Also it is observed that a Fredholm integral equation always transforms into a boundary value problem.

Books Suggested:

1. Jerri, A.J., Introduction to Integral Equations with Applications, A Wiley-Interscience Publication, 1999.
2. Kanwal, R.P., Linear Integral Equations, Theory and Techniques, Academic Press, New York.
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